ELSEVIER Journal of Geometry and Physics 25 (1998) 46–68

JOURNAL OF GEOMETRY AND PHYSICS

Involutive braided $Spin(4 - h, h)$ groups

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Received 20 November 1996; received in revised form 27 February 1997

Abstract

Using as a starting point a deformed Clifford algebra with an involutive braid we introduce a natural deformation of the Cartan procedure to construct $Spin(4 - h, h)$ groups. The method presented produces braided groups instead of quantum groups. We also study the induced left comodule structure of the spinor bialgebra. Finally, we construct the braided special orthogonal group and we establish the comodule homomorphism between the braided special orthogonal groups and the respective braided spin matrices. © 1998 Elsevier Science B.V.

Subj. Class.: Quantum groups
1991 MSC: 20F36, 53A50
Keywords: Deformed Clifford algebra; Involutive braid; Braided group; Spinor bialgebra

1. Introduction

Since the discovery of Quantum Groups, many authors have considered the possibility of q deforming the Lorentz and Poincaré groups as part of a more general program of studying quantum deformed field theories. There have been various approaches published in the literature (see e.g. [1–4]). One of these approaches, due to Lukierski et al. [1], is based on the standard theory of quantum groups and a particular deformation consisting on the assumption that the space coordinates commute among themselves, while the time coordinate does not commute with the rest. On the other hand, the deformation used in [2] makes use of a Hecke braid for the space–time coordinates, which leads to a differential calculus involving nonlinear operators as well as an extra generator of dilations. As an alternative to avoid these later problems an involutive braid is used in [3]. Using as a starting point a deformed Clifford algebra with an involutive braid also, the authors [5] have derived generalizations to

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PII S0393-0440(97)00021-1

$2n$-dimensions of the quantum Poincaré and Spin groups, which also imply a more simple q-differential calculus and no dilatations. A rather different approach is that followed by Majid [4], based on a generalization of quantum groups, the so-called 'braided groups', which use a deformed tensor product that results in a braided Hopf algebra. Introducing a braided coaddition on vectors to induce the translations of the Poincaré group, Majid shows that several q-deformed Poincaré groups have the structure of a semi-direct product and coproduct $B \bowtie SO_q(1,3)$, where B has a braided-group structure on the q-Minkowski space.

The purpose of this paper is to show that the natural deformation of the classical Cartan approach to construct spin groups [6] produces braided spin groups. In consequence the q-deformed Lorentz group constructed from the braided spin group is also a braided group. In this article we use an involutive braiding, so our braided groups belong to a monoidal category of the Mac Lane type [7]. It is possible to generalize our results to the case of more general braidings, such as the Hecke braiding. However, in these more general cases the differential calculus associated to the quantum plane is compatible with the *-structure, for pseudo-euclidean signatures, only for nonlinear differential operators.

The paper is organized in the following way. First, in order to make our presentation more self-contained, we review in Section 2 some of the basic axioms related to braided groups; while in Section 3 we include the most relevant aspects of the construction of a quantum Clifford algebra, with an involutive braid and a compatible *-structure, which we have developed in full detail elsewhere [5]. By requiring that the fundamental property of spinor transformations be preserved in the braided case, a non-commutative algebra is induced for the "coordinates" of the underlying Euclidean or pseudo-Euclidean spaces. From this a non-commutative algebra is generated for the fundamental "spinors". In Section 4 we introduce the natural deformation of the Cartan procedure to construct Spin groups. The method presented produces the braided $Spin(4-h,h)$ groups. We also study in this section the induced left comodule structure of the spinor bialgebra, and the consistency of the coproduct and coaction maps on it. Finally, Section 5 is devoted to the construction of the braided special orthogonal group, and to a discussion of the comodule homomorphism between the braided special orthogonal groups and the respective braided spin matrices derived in Section 4.

2. Braided groups

We review in this section those essential categorical concepts of the formalism of braided groups which we shall be needing for our discussion in latter sections. For a more detailed analysis of this subject we refer the reader to the material contained in [4,8,9].

Braided categories arose naturally from knot theory. They provide a formalism for generalizing supersymmetry and quantum groups, with the hope of achieving a systematic approach to q-deforming structures in physics [4,8]. In the formalism of braided geometry, vector spaces, linear maps and tensor products of linear algebras, on which most of the mathematics used in physics is based, are replaced by a new category which mimics, by axiomatization, most of the properties of the *Vect* category. The main axiom which is changed is that in a general category with product, one postulates the existence of a natural transformation Ψ, between the two functors $(C, B) \rightsquigarrow C \otimes B$ and $B \otimes C$ from $\mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$. Thus one has a braided tensor product algebra $(\cdot_B \otimes \cdot_C)\Psi_{C,B}$. Here the *braiding* $\Psi_{C,B}$: $C \otimes B \to B \otimes C$ is a collection of isomorphisms that expresses the degree of commutativity of the algebra structure on the tensor product, i.e., we deform $(a \otimes c)(b \otimes c) = ab \otimes cd$ to

$$(a \otimes c) \cdot_\Psi (b \otimes d) = a\Psi(c \otimes b)d, \tag{1}$$

where, the braiding Ψ satisfies the braid relation

$$(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi) = (\Psi \otimes \mathrm{id})(\mathrm{id} \otimes \Psi)(\Psi \otimes \mathrm{id}). \tag{2}$$

More formally, let $\mathfrak{A}$ be an associative algebra with unit $1 \in \mathfrak{A}$, and with multiplication map $m : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$. We assume that $\mathfrak{A}$ is endowed with a structure of a coalgebra, given by a coproduct $\phi : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ and a counit $\epsilon : \mathfrak{A} \to \mathbb{C}$. Furthermore there exists a bijective linear map $\kappa : \mathfrak{A} \to \mathfrak{A}$. We then have the following:

Definition. A braided group or braided Hopf algebra, is a pair $\mathfrak{G} = (\mathfrak{A}, \{\phi, e, \kappa, \Psi\})$ satisfying the commutative diagrams

$$\begin{array}{ccc} \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)} & \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} \\ {\scriptstyle m\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes m} \\ \mathfrak{A} \otimes \mathfrak{A} & \underset{\Psi}{\longrightarrow} & \mathfrak{A} \otimes \mathfrak{A}, \end{array} \tag{3}$$

$$\begin{array}{ccc} \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{(\mathrm{id}\otimes\Psi)(\Psi\otimes\mathrm{id})} & \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} \\ {\scriptstyle \mathrm{id}\otimes m}\big\downarrow & & \big\downarrow{\scriptstyle m\otimes\mathrm{id}} \\ \mathfrak{A} \otimes \mathfrak{A} & \underset{\Psi}{\longrightarrow} & \mathfrak{A} \otimes \mathfrak{A}, \end{array} \tag{4}$$

$$\begin{array}{ccc} \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{\Psi} & \mathfrak{A} \otimes \mathfrak{A} \\ {\scriptstyle \phi\otimes\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}\otimes\phi} \\ \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow[(\Psi\otimes\mathrm{id})(\mathrm{id}\otimes\Psi)]{} & \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}, \end{array} \tag{5}$$

$$\begin{array}{ccc} \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow{\Psi} & \mathfrak{A} \otimes \mathfrak{A} \\ {\scriptstyle \mathrm{id}\otimes\phi}\big\downarrow & & \big\downarrow{\scriptstyle \phi\otimes\mathrm{id}} \\ \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A} & \xrightarrow[(\mathrm{id}\otimes\Psi)(\Psi\otimes\mathrm{id})]{} & \mathfrak{A} \otimes \mathfrak{A} \otimes \mathfrak{A}, \end{array} \tag{6}$$

together with the antipode axiom

$$1\epsilon = m(\mathrm{id} \otimes \kappa)\phi = m(\kappa \otimes \mathrm{id})\phi, \tag{7}$$

and the multiplicativity axiom of the coproduct, $\phi(ab) = \phi(a)\phi(b)$. Relations (3) and (4) ensure that Ψ defines an associative algebra structure on $\mathfrak{A} \otimes \mathfrak{A}$, such that $1 \otimes 1$ is the unit element. Also, instead of the usual multiplication anti-homomorphism of the antipode, we have

$$\kappa(ab) = m\Psi(\kappa \otimes \kappa)(a \otimes b), \tag{8}$$

and instead of the usual anti-cohomomorphism we need

$$\phi \circ \kappa = (\kappa \otimes \kappa) \circ \Psi\phi. \tag{9}$$

From here on, we shall denote by $B\underline{\otimes}B$ the braided product with tensor multiplication given by (1).

Some braided groups have, in addition, a quasi-triangular structure if there exists a universal braid matrix, $R = \sum_i \alpha_i \otimes \beta_i \in B \otimes B$, such that

$$\pi(\phi(a))R = R\phi(a), \quad (\phi \otimes I)R = R^{13}R^{23}, \quad (I \otimes \phi)R = R^{13}R^{12}, \tag{10}$$

where $R^{13} = \alpha_i \otimes 1 \otimes \beta_i$, $R^{23} = 1 \otimes \alpha_i \otimes \beta_i$, $R^{12} = \alpha_i \otimes \beta_i \otimes 1 \in B \otimes B \otimes B$, with $\pi : B \otimes B \to B \otimes B$ the standard transposition.

From the above axioms it is possible to obtain the quantum Yang–Baxter equation in the form $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$. Quantum groups are a special case of the braided groups when the quasi-triangular structure is satisfied, and the tensor product is the usual one, i.e., when Ψ and π are the same in the preceding relations.

Following [9,10] we introduce an additional *-structure on the braided Hopf algebra, as an involutive *anti-multiplicative* map $* : \mathfrak{A} \to \mathfrak{A}$, compatible with the coproduct. Then we have:

Definition. An involutive antilinear map $* : \mathfrak{A} \to \mathfrak{A}$ is called a *-structure on $\mathfrak{B}$ iff
(i) $*m = m\pi(* \otimes *)$,
(ii) $\phi* = (* \otimes *)\pi\Psi^{-1}\phi$.

In this way $(\mathfrak{A}, *)$ becomes a *-algebra.

3. Clifford, space–time, and spinor algebras

A general theory of deformed Clifford algebras was developed in [11], based on a quantum generalization of Cartan's theory of spinors. By particularizing this theory to the case of multiparametric involutive braids, we obtained in [5] that the generators of the deformed Clifford algebra $Cl(\tau, W)$ satisfy the relations:

$$\begin{aligned} &H(e_i)H(e_j) + \sum_{k,l} \tau_{ij}{}^{kl} H(e_k)H(e_l) = 0, \\ &H(e'_i)H(e'_j) + \sum_{k,l} \tau_{ij}{}^{kl} H(e'_k)H(e'_l) = 0, \\ &H(e'_i)H(e_j) + \sum_{k,l} \bar{\tau}_{ij}{}^{kl} H(e_k)H(e'_l) = \delta_{ij}E, \end{aligned} \tag{11}$$

$$H(e_i)H(e'_j) + \sum_{k,l} \bar{\tau}_{ij}{}^{kl} H(e'_k)H(e_l) = \eta_{ij} E,$$

where $W = V \oplus V'$ is the underlying 2ν-dimensional (pseudo)-Euclidean space, V and its dual V' are isotropic subspaces, both of dimension ν, and $\{e_i\}$, $\{e'_i\}$ are their respective bases.

The matrix elements $\tau_{ij}{}^{kl}$ and $\bar{\tau}_{ij}{}^{kl}$ in (11) are determined by the automorphisms

$$\begin{aligned} &\tau(e_i \otimes e_j) = \mu_{ij}^{-1}(e_j \otimes e_i), \quad i < j, \\ &\tau(e_i \otimes e_i) = e_i \otimes e_i, \\ &\tau(e_i \otimes e_j) = \mu_{ij}(e_j \otimes e_i), \quad i > j, \end{aligned} \tag{12}$$

$$\begin{aligned} &\bar{\tau}(e'_i \otimes e_j) = \mu_{ij}(e_j \otimes e'_i), \quad i < j, \\ &\bar{\tau}(e'_i \otimes e_i) = e_i \otimes e'_i, \\ &\bar{\tau}(e'_i \otimes e_j) = \mu_{ij}^{-1}(e_j \otimes e'_i), \quad i > j, \end{aligned} \tag{13}$$

such that

$$\begin{aligned} &\mu_{ij} = \mu_{ji}^{-1} = \exp(\mathrm{i}\theta_k), \quad 1 \le k \le \nu - h - 1 \ \text{ for } i, j \le \nu - h, \\ &\mu_{ij} = \mu_{ji}^{-1} = \exp(\mathrm{i}\lambda_k), \quad 1 \le k \le h - 1 \ \text{ for } i, j \le \nu - h + 1, \\ &\mu_{ij} = \mu_{ji}^{-1} = \mu_k \in \mathbb{R}, \quad 1 \le k \le \nu - h \ \text{ for } \begin{cases} i \le \nu - h, & j \ge \nu - h + 1, \\ j \le \nu - h, & i \ge \nu - h + 1. \end{cases} \end{aligned} \tag{14}$$

Requiring that the fundamental property of spinor transformations be preserved in the quantum case, a non-commutative algebra $\mathcal{A}$ was obtained for the *coordinates* of the underlying pseudo-Euclidean spaces given by

$$\begin{aligned} x^i x^j &= m\bar{\tau}(x^i \otimes x^j) = \mu_{ij} x^j x^i, \\ x'^i x^j &= m\tau(x'^i \otimes x^j) = \mu_{ij}^{-1} x^j x'^i, \\ x^i x'^j &= m\tau(x^i \otimes x'^j) = \mu_{ij}^{-1} x'^j x^i, \\ x'^i x'^j &= m\bar{\tau}(x'^i \otimes x'^j) = \mu_{ij} x'^j x'^i. \end{aligned} \tag{15}$$

This algebra has a consistent anti-multiplicative *-structure, inherited from the Clifford algebra, such that

$$(* \otimes *)\pi \hat{R} = \hat{R}(* \otimes *)\pi. \tag{16}$$

In the above, π is the standard permutation operator, and $\hat{R}$ is a homomorphism $\hat{R} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ which satisfies the braid relation (2). In block matrix notation it is given by

$$\hat{R} = \begin{pmatrix} \bar{\tau} & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & 0 & 0 & \bar{\tau} \end{pmatrix}, \tag{17}$$

where τ and $\bar{\tau}$ are the braid isomorphisms defined in (12) and (13).

The quotient algebra $\mathcal{A}_{\hat{R}} = \mathcal{A}/\mathcal{I}_{\hat{R}}$, where $\mathcal{I}_{\hat{R}}$ is the two-sided ideal in $\mathcal{A}$ generated by $(1-\hat{R})(x\otimes x)$, is an algebra of functions on the quantum n-dimensional vector space associated with the matrix $\hat{R}$. Furthermore, with a left coaction map $\delta : \mathcal{A}_{\hat{R}} \to \mathcal{T}_R \otimes \mathcal{A}_{\hat{R}}$, together with the requirement of invariance of the fundamental quadric $\langle \mathbf{x}, \mathbf{x}\rangle$, the algebra $\mathcal{A}_{\hat{R}}$ acquires a comodule structure for the quantum matrix algebra $\mathcal{T}_R$ of the $SO_q(2\nu-h, h)$ groups (cf. [5]).

3.1. q-Spinors

Recalling now that, in analogy with the classical case, the q-Clifford product is uniquely determined by the relations

$$H_i \cdot 1 = e_i, \qquad H_i \cdot e_j = e_i \wedge e_j, \qquad H_i' \cdot e_j = \iota_{e_i'} e_j = e_i'(e_j) = \delta_{ij}, \tag{18}$$

where the wedge product involves the appropriate braiding given in Eq. (12), we can define a q-spinor by

$$\xi = \sum_{p=0}^{\nu} \sum_{k_1<\cdots<k_p} \xi^{k_1\cdots k_p} H_{k_1} \cdots H_{k_p} \cdot 1. \tag{19}$$

Here the 2^ν components $\xi^{k_1\cdots k_p}$ are the generators of a non-commutative free algebra Ξ, and the symbol $\sum_{k_1<\cdots<k_p}$ is to be interpreted as no sum in the case $p=0$, so $\xi^{k_1\cdots k_p} = \xi^0$ when $p=0$.

Note that in the classical limit $\mu_{ij} \to 1$, the above expression reduces to the usual definition of a spinor as an element in the graded Grassmann algebra of the basis vectors in the corresponding (pseudo)-Euclidean space to which the spinor is associated.

We can introduce a bilinear inner product on the q-spinor vector spaces $\mathcal{S}$ spanned by the generators of Ξ by first defining the involutive and anti-multiplicative $\top$-transpose operation, $\xi \in \mathcal{S} \to \xi^\top \in \mathcal{S}'$, which maps linearly spinors in $\mathcal{S}$ to spinors in the dual space $\mathcal{S}'$. This operation is uniquely defined by its action on the generators of the Clifford algebra:

$$(H_i \cdot 1)^\top = 1' \cdot H_i', \qquad (H_i H_j \cdot 1)^\top = 1' \cdot H_j' \cdot H_i'. \tag{20}$$

Thus the $\top$-transpose operation maps Clifford product action from the left to Clifford product action from the right, and

$$\xi^\top = \sum_{p=0}^{\nu} \sum_{k_1<\cdots<k_p} \xi^{k_1\cdots k_p} \otimes 1' \cdot H_{k_p}' \cdots H_{k_1}'. \tag{21}$$

Note that by virtue of (18) and (20) the elements $\{(H_{k_1} \cdots H_{k_p} \cdot 1)^\top = 1' \cdot H_{k_p}' \cdots H_{k_1}'\}$ form a basis reciprocal to $\{H_{k_1} \cdots H_{k_p} \cdot 1\}$, $k_1 < \cdots < k_p$, under the scalar product

$$[1' \cdot H_{k_p}' \cdots H_{k_1}', H_{k_1} \cdots H_{k_p} \cdot 1] = 1' \cdot H_{k_p}' \cdots H_{k_1}' \cdot H_{k_1} \cdots H_{k_p} \cdot 1 = 1'(1) = 1. \tag{22}$$

This allows us to define a scalar a fundamental spinor bilinear, as in [5], by means of

$$(\xi, \eta) = [\xi^{\top}, C \cdot \eta]. \tag{23}$$

where C is a spinor metric operator given by

$$C = \sum_{p=0}^{\nu} (-1)^{(\nu-p)(\nu-p+1)/2} \sum_{\substack{\pi \in S_\nu \\ \pi(1)<\cdots<\pi(p) \\ \pi(p+1)<\cdots<\pi(\nu)}} (-1)^{l(\pi)} a_{\pi(1)\cdots\pi(p)}(\mu) \times H_{\pi(1)} \cdots H_{\pi(p)} (H_{\pi(p+1)} \cdots H_{\pi(\nu)})^{\top} \tag{24}$$

with $l(\pi)$ = length of the permutation π, and

$$a_{\pi(1)\cdots\pi(p)} = [\mu_{\langle\pi(1)\pi(\nu)\rangle} \cdots \mu_{\langle\pi(1)\pi(p+1)\rangle} \cdots \mu_{\langle\pi(p)\pi(\nu)\rangle} \cdots \mu_{\langle\pi(p)\pi(p+1)\rangle}]^{1/2},$$
$$a_{\pi(1)\cdots\pi(\nu)} = a_0 = 1, \tag{25}$$

and the symbol $\langle\ \rangle$ denotes pair ordering of indices so that the first one is lower than the second. It is easy to verify from (24) that

$$C^{\top} = (-1)^{\nu(\nu+1)/2} C. \tag{26}$$

Making use of (24) it can be readily shown that (23) may be written as

$$(\xi, \eta) = \sum_{p=0}^{\nu} (-1)^{(\nu-p)(\nu-p+1)/2} \times \sum_{\substack{\pi(1)<\cdots<\pi(p) \\ \pi(p+1)<\cdots<\pi(\nu)}} (-1)^{l(\pi)} a_{\pi(1)\ldots\pi(p)} \xi^{\pi(1)\ldots\pi(p)} \eta^{\pi(p+1)\ldots\pi(\nu)}. \tag{27}$$

Now, taking as a basis the 2^ν generators of the Clifford algebra $\{1, H_{k_1} \cdots H_{k_p}.1 \mid k_1 < k_2 \cdots < k_p, p = 1, 2, \ldots, \nu\}$ ordered in such a way that those with an even number of indices and in an increasing degree sequence come first, followed by those elements with an odd number of indices also in an increasing degree sequence, we can write a spinor as a column vector where the first $2^{\nu-1}$ entries correspond to a semi-spinor of the first type (which we shall denote by φ), while the last $2^{\nu-1}$ entries correspond to a semi-spinor of the second type (which we shall denote by ψ), in Cartan's terminology.

It is evident from (27) that the fundamental spinor bilinear involves products of components of semi-spinors of the same type if ν = even while if ν = odd the products are of semi-spinors of the two different types.

Based on the fact that the total number, $2^{2\nu}$, of products of the components of two spinors equals the sum of the degrees of irreducible tensors found, a classical theorem in spinor calculus (cf. [6]) states that a spinor bilinear is completely reducible with respect to the group of rotations and reversals and decomposes into a scalar, a vector, a bivector, ..., an n-vector.

We shall make use of this theorem to obtain commutation relations for the free algebra $\mathfrak{S}$ of q-spinors. Thus, guided by the fact that in the limit $\mu \to 1$ we must obtain the classical expression for the components of a vector, written as a spinor bilinear formed with the semi-spinor components of the two spinors ξ, $\tilde{\xi}$ we take as an ansatz, for $\nu = 2$ (which is the case we shall be considering in this paper), and for an isotropic basis,

$$x^1 = \tfrac{1}{2}(\psi^1\tilde{\varphi}^2 + \tilde{\psi}^1\varphi^2), \tag{28}$$

$$x'^1 = -\tfrac{1}{2}(\psi^2\tilde{\varphi}^1 + \tilde{\psi}^2\varphi^1), \tag{29}$$

$$x^2 = \tfrac{1}{2}(\psi^2\tilde{\varphi}^2 + \tilde{\psi}^2\varphi^2), \tag{30}$$

$$x'^2 = \tfrac{1}{2}\mu(\psi^1\tilde{\varphi}^1 + \tilde{\psi}^1\varphi^1). \tag{31}$$

Furthermore, from (24) and (25), we have that

$$C = H_1 H_2 - \sqrt{\mu} H_1 H_2' + \sqrt{\mu} H_2 H_1' - H_2' H_1', \tag{32}$$

and from (19), making use of the spinor ordering described above, we can write

$$\xi = \varphi^1 + \psi^1 H_1 + \psi^2 H_2 + \varphi^2 H_1 H_2 \tag{33}$$

and

$$\xi^\top C\tilde{\xi} = \varphi^2\tilde{\varphi}^1 - \sqrt{\mu}\psi^1\tilde{\psi}^2 + \sqrt{\mu}\psi^2\tilde{\psi}^1 - \varphi^1\tilde{\varphi}^2. \tag{34}$$

Thus

$$\begin{aligned} CH(\mathbf{x})\tilde{\xi} = {} & x'^1\tilde{\psi}^1 H_1 H_2 \cdot 1 + x'^2\tilde{\psi}^2 H_1 H_2 \cdot 1 - \sqrt{\mu}x^2\tilde{\varphi}^1 H_1 \cdot 1 - \sqrt{\mu}x'^1\tilde{\varphi}^2 H_1 \cdot 1 \\ & + \sqrt{\mu}x^1\tilde{\varphi}^1 H_2 \cdot 1 - \frac{1}{\sqrt{\mu}}x'^2\tilde{\varphi}^2 H_2 \cdot 1 - x^1\tilde{\psi}^2 + \mu x^2\tilde{\psi}^1 \end{aligned} \tag{35}$$

and

$$\begin{aligned} (H(\mathbf{x})\xi)^\top = {} & (x^1\varphi^1)^\top 1' \cdot H_1' + (x^1\psi^2)^\top 1' \cdot H_2' \cdot H_1' + (x^2\varphi^1)^\top 1' \cdot H_2' \\ & + (x^2\psi^1)^\top 1' \cdot H_1' \cdot H_2' + (x'^1\psi^1)^\top 1' + (x'^1\varphi^2)^\top 1 \cdot H_2' \\ & + (x'^2\psi^2)^\top 1' - \frac{1}{\sqrt{\mu}}(x'^2\varphi^2)^\top 1' \cdot H_1'. \end{aligned} \tag{36}$$

Denoting by Ψ the braid operator between the coordinates and the spinor components, we have

$$(x^\alpha\xi^\beta)^\top = m\Psi(x^\alpha \otimes \xi^\beta). \tag{37}$$

We now require that

$$(H(\mathbf{x})\xi, H(\mathbf{x})\tilde{\xi}) = (\xi, \tilde{\xi}), \tag{38}$$

as in the classical case, and we further make use of (35), (36) and (34), together with the fundamental property of spinor transformations

$$H(\mathbf{x})H(\mathbf{x}) = \langle\mathbf{x}, \mathbf{x}\rangle E, \tag{39}$$

where

$$\langle \mathbf{x}, \mathbf{x}\rangle = x'^1 x^1 + x'^2 x^2. \tag{40}$$

It follows then that

$$\begin{aligned}
x^1\varphi^1 &= m\Psi(x^1\otimes\varphi^1) = \frac{1}{\sqrt{\mu}}\varphi^1 x^1, \quad x^1\psi^2 = m\Psi(x^1\otimes\psi^2) = \sqrt{\mu}\psi^2 x^1,\\
x^2\varphi^1 &= m\Psi(x^2\otimes\varphi^1) = \sqrt{\mu}\varphi^1 x^2, \quad x^2\psi^1 = m\Psi(x^2\otimes\psi^1) = \frac{1}{\sqrt{\mu}}\psi^1 x^2,\\
x'^1\varphi^2 &= m\Psi(x'^1\otimes\varphi^2) = \frac{1}{\sqrt{\mu}}\varphi^2 x'^1, \quad x'^1\psi^1 = m\Psi(x'^1\otimes\psi^1) = \sqrt{\mu}\psi^1 x'^1,\\
x'^2\varphi^2 &= m\Psi(x'^2\otimes\varphi^2) = \sqrt{\mu}\varphi^2 x'^2, \quad x'^2\psi^2 = m\Psi(x'^2\otimes\psi^2) = \frac{1}{\sqrt{\mu}}\psi^2 x'^2.
\end{aligned} \tag{41}$$

Using (28)–(31) in the above relations, it is straightforward to show that

$$\begin{aligned}
&\varphi^1\varphi^2 = \varphi^2\varphi^1, \quad \varphi^1\psi^1 = \sqrt{\mu}\psi^1\varphi^1, \quad \varphi^1\psi^2 = \frac{1}{\sqrt{\mu}}\psi^2\varphi^1,\\
&\psi^1\psi^2 = \psi^2\psi^1, \quad \varphi^2\psi^2 = \sqrt{\mu}\psi^2\varphi^2, \quad \varphi^2\psi^1 = \frac{1}{\sqrt{\mu}}\psi^1\varphi^2.
\end{aligned} \tag{42}$$

In this way the spinor algebra $\mathfrak{S}$ becomes a factor algebra $\mathfrak{S}_B = \mathfrak{S}/I_B$ with I_B being the two-sided ideal generated by (42). In the following section we shall show that the generators of this factor algebra acquire the structure of a comodule vector space for the braided spin group.

4. Braided spin groups

In the classical Cartan spinor theory, the action of the operator $H(\mathbf{x}) = \sum_{i=1}^{\nu}(x^i H_i + x'^i H'_i)$ on spinors, with $\mathbf{x}$ a unit vector, corresponds to a reflection in the hyperplane perpendicular to $\mathbf{x}$. A proper rotation on spinors then corresponds to an even product of Clifford operators.

Thus, because of the isomorphism between $Cl(W)$ and $H(W)$, the $Pin(2\nu)$ group, associated with an underlying 2ν-dimensional (pseudo)-Euclidean space, is given by the set $Pin(2\nu) = \{s \in Cl(W), \dim(W) = 2\nu | s = H(\mathbf{x}_1)\cdots H(\mathbf{x}_k), k = 1, \ldots, 2\nu, \mathbf{x}_i = \mathbf{x}_i^* \in W, |\mathbf{x}_i| = 1\}$. Taking the even part of the Clifford algebra we have $Spin(2\nu) = \{s \in Pin(2\nu), \text{ with } s = H(x_1)\cdots H(x_{2k})\}$.

Consider now the element $s = H(x_1)\cdots H(x_{2k})$. Since the Clifford algebra is associative, we can group the above product in pairs so that

$$s = (H(\mathbf{x}_1)H(\mathbf{x}_2))\cdots(H(\mathbf{x}_{2k-1})H(\mathbf{x}_{2k})).$$

But each pair $B(\mathbf{x}_i, \mathbf{x}_{i+1})$, with i odd, has matrix representation

$$\begin{pmatrix} B_1^i & 0 \\ 0 & B_2^i \end{pmatrix} \in SL(2^{\nu-1}, \mathbb{C}) \times SL(2^{\nu-1}, \mathbb{C}).$$

Hence s has the matrix representation

$$s = \begin{pmatrix} \prod_i B_1^i & 0 \\ 0 & \prod_i B_2^i \end{pmatrix} \in SL(2^{\nu-1}, \mathbb{C}) \times SL(2^{\nu-1}, \mathbb{C}).$$

This is due to the fact that $\det(B_j^1 \dots B_j^k) = \prod_i \det(B_j^i) = 1$, for $j = 1, 2$.

Consequently in the classical case, the quadratic algebra of the Clifford generators is the essential building block for the *Spin* groups, and is the basis of Cartan's construction of the double covering of the (pseudo)-orthogonal groups.

It is the purpose of this section to show that a similar approach may be followed to obtain the braided $Spin(4-h, h)$ groups, as a braided Hopf algebra of polynomial functions in the generators made out of the non-commutative entries of matrix representations of the dyadic operators $B(\mathbf{x}_i, \mathbf{x}_{i+1})$. The extension of this procedure to higher-dimensional spaces remains to be investigated.

We thus begin by considering the operator $B(\mathbf{x}, \mathbf{y}) = H(\mathbf{x})H(\mathbf{y})$, where the components $\{x^1, x^2, x'^1, x'^2\}$ and $\{y^1, y^2, y'^1, y'^2\}$ (in an isotropic basis) are no longer commutative, i.e., we shall assume that the *coordinates* of the unit vectors $\mathbf{x}$ and $\mathbf{y}$ are required to satisfy the commutation relations (15), extended to apply to *coordinates* of different vectors also. Note that we can still assume $|\mathbf{x}| = |\mathbf{y}| = 1$, because these products are central to $\mathcal{A}_{\hat{R}}$. We shall denote by $\mathcal{B}$ the algebra generated by the matrix elements of the operator $B(\mathbf{x}, \mathbf{y})$.

As we mentioned at the end of the preceding section, we can give to the vector space $\mathcal{S}$ of the generators of the factor algebra $\mathfrak{S}_B$ a comodule structure by introducing the coaction map $\delta : \mathcal{S} \to \mathcal{B} \otimes \mathcal{S}$, where $B(\mathbf{x}, \mathbf{y})$ acts on $\mathcal{S}$ through Clifford multiplication. Thus,

$$\begin{aligned} B(\mathbf{x}, \mathbf{y}) \dot{\otimes} \xi = \sum_{p=0}^{2} \sum_{i,j=1}^{2} \sum_{k_1 < \cdots < k_p} [& x^i y^j \otimes \xi^{k_1 \cdots k_p} H_i H_j H_{k_1} \cdots H_{k_p} \cdot 1 \\ & + x'^i y^j \otimes \xi^{k_1 \cdots k_p} H'_i H_j H_{k_1} \cdots H_{k_p} \cdot 1 \\ & + x^i y'^j \otimes \xi^{k_1 \cdots k_p} H_i H'_j H_{k_1} \cdots H_{k_p} \cdot 1 \\ & + x'^i y'^j \otimes \xi^{k_1 \cdots k_p} H'_i H'_j H_{k_1} \cdots H_{k_p} \cdot 1]. \end{aligned} \tag{43}$$

Taking now as a basis the four elements $\{1, H_{k_1} H_{k_2} \mid k_1 < k_2\}$, and applying the Cartan ordering procedure in terms of semi-spinors of the first type followed by semi-spinors of the second type, we can recombine the coefficients in (43) (making use of our q-Clifford algebra) as new factors in such a basis. Thus we can rewrite (43) as $\sum_{\beta=1}^{4} b^\alpha{}_\beta \otimes \xi^\beta$, $\alpha, \beta = 1, \dots, 4$, where the first two entries in the column ξ^β correspond to a semi-spinor of the first type, while the last two entries correspond to a semi-spinor of the second type. The rearranged coefficients in (43) yield the entries $b^\alpha{}_\beta$ of the block-diagonal matrix representation of $B(\mathbf{x}, \mathbf{y})$, from which the free algebra $\mathcal{B}$ of non-commutative polynomials is generated. Furthermore, the algebra (15) of the "coordinates" which occur in $b^\alpha{}_\beta$, determines the commutation relations for the latter.

Thus,

$$B \otimes \xi = (b^\alpha{}_\beta) \otimes \xi^\beta, \tag{44}$$

where

$$
(\xi) = \begin{pmatrix} \xi^0 \\ \xi^{12} \\ \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} \varphi^1 \\ \varphi^2 \\ \psi^1 \\ \psi^2 \end{pmatrix}, \tag{45}
$$

and the entries of the block-diagonal matrix $(b^\alpha{}_\beta)$, are given by:

$$
\begin{aligned}
b^1{}_1 &= x'^1 y^1 + x'^2 y^2, & b^1{}_2 &= x'^2 y'^1 - \mu^{-1} x'^1 y'^2, \\
b^2{}_1 &= x^1 y^2 - \mu x^2 y^1, & b^2{}_2 &= x^1 y'^1 + x^2 y'^2, \\
b^3{}_3 &= x^1 y'^1 + x'^2 y^2, & b^3{}_4 &= x^1 y'^2 - \mu^{-1} x'^2 y^1, \\
b^4{}_3 &= x^2 y'^1 - \mu x'^1 y^2, & b^4{}_4 &= x'^1 y^1 + x^2 y'^2, \\
b^i{}_{j+2} &= 0, \quad b^{i+2}{}_j = 0, & & i, j = 1, 2.
\end{aligned} \tag{46}
$$

Moreover, making use of (15), it immediately follows that

$$
\begin{aligned}
b^1{}_2 b^3{}_4 &= \mu^2 b^3{}_4 b^1{}_2, \qquad b^1{}_2 b^4{}_3 = \mu^{-2} b^4{}_3 b^1{}_2, \\
b^2{}_1 b^3{}_4 &= \mu^{-2} b^3{}_4 b^2{}_1, \qquad b^2{}_1 b^4{}_3 = \mu^2 b^4{}_3 b^2{}_1, \\
[b^i{}_j, b^l{}_m] &= 0, \quad [b^{i+2}{}_{j+2}, b^{l+2}{}_{m+2}] = 0, \qquad i, j, l, m = 1, 2.
\end{aligned} \tag{47}
$$

Note from the above that elements in the same 2×2 block matrix commute with each other. Also note that the matrix components $b^\alpha{}_\beta$ inherit the *-structure for the *coordinates* given by (cf. [5])

$$
\begin{aligned}
(x^i)^* &= x'^i, \quad i = 1, \ldots, 2-h, \qquad (x'^i)^* = x^i, \quad i = 1, \ldots, 2-h, \\
(x^i)^* &= x^i, \quad i = 2-h+1, \ldots, 2, \qquad (x'^i)^* = x'^i, \quad i = 2-h+1, \ldots, 2.
\end{aligned} \tag{48}
$$

Explicitly, for $h = 1$ we have $\mu \in \mathbb{R}$, and

$$
(b^1{}_1)^* = b^3{}_3, \quad (b^1{}_2)^* = -b^3{}_4, \quad (b^2{}_1)^* = -b^4{}_3, \quad (b^2{}_2)^* = b^4{}_4; \tag{49}
$$

and for $h = 0$

$$
\mu^* = \mu^{-1}, \tag{50}
$$

$$
(b^1{}_1)^* = b^2{}_2, \quad (b^1{}_2)^* = -b^2{}_1, \quad (b^3{}_3)^* = b^4{}_4, \quad (b^3{}_4)^* = -b^4{}_3. \tag{51}
$$

It is easy to verify that this induced *-structure is compatible with (47), and that for Minkowski space ($h = 1$), it interchanges the matrix elements from the upper block with those from the lower block; while for Euclidean space ($h = 0$) the *-operation is closed within each block and corresponds to the matrix representation of the group $SU(2)$.

As a next step we need to show that the algebra $\mathcal{B}$, generated by the matrix elements $b^\alpha{}_\beta$, has a natural braided bialgebra structure. With this purpose let I_Ψ be the two-sided ideal of $\mathcal{B}$ generated by (47) (note that the braid operator Ψ is induced by the previously obtained braid operator for the *coordinates*). The quotient algebra $\mathcal{B}_\Psi = \mathcal{B}/I_\Psi$ has the structure of a braided group, as defined in Section 2. Indeed, the coalgebra structure is given by a coproduct and counit defined by

$$\phi(b^{\alpha}{}_{\beta}) = b^{\alpha}{}_{\gamma} \otimes b^{\gamma}{}_{\beta}, \epsilon(b^{\alpha}{}_{\beta}) = \delta^{i}_{j}. \tag{52}$$

Because of the braided algebra structure, these maps can be seen to be compatible with (47), after recalling that

$$\phi(ab) = \phi(a)\phi(b), \qquad \epsilon(ab) = \epsilon(a)\epsilon(b) \tag{53}$$

with

$$(a \otimes c)(b \otimes d) = a\Psi(c \otimes b)d. \tag{54}$$

It is easy to verify that $\Psi = \Psi^{-1}$ and also that the functoriality relations (3)–(6), and the braid relation (2), given in Section 2, are satisfied. Thus Ψ is an involutive braid operator. Finally, the antipode is clearly given by $B(\mathbf{y}, \mathbf{x})$.

So far we have obtained the braided group related to a deformation of the group $GL(4, \mathbb{C})$. In fact, as we shall show next, our braided group is smaller and corresponds to a *subgroup* of the braided group $SL(4, \mathbb{C})$. To be more precise, it is the deformed $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ group.

4.1. The braided determinant of the group

Since the matrix $(b^{\alpha}{}_{\beta})$ is block diagonal, the coaction map δ on the spinor space can be decomposed into separate coactions on the semi-spinors of the first and second types. Thus,

$$\delta_1 : \mathcal{S}_1 \to \mathcal{B}_{\Psi_1} \otimes \mathcal{S}_1, \qquad \delta_2 : \mathcal{S}_2 \to \mathcal{B}_{\Psi_2} \otimes \mathcal{S}_2 \tag{55}$$

with

$$\begin{aligned} \delta_1(\varphi^i) &= \sum_{j=1,2} b^i{}_j \otimes \varphi^j, \quad i = 1, 2, \\ \delta_2(\psi^i) &= \sum_{j=1,2} b^{i+2}{}_{j+2} \otimes \psi^j, \quad i = 1, 2. \end{aligned} \tag{56}$$

Here $\mathcal{S}_1$ is the vector space generated by the spinorial components $\{\varphi^i\}_{i=1,2}$; and $\mathcal{S}_2$ is the vector space generated by $\{\psi^i\}_{i=1,2}$; while $\mathcal{B}_{\Psi_1} = \mathcal{B}_1/I_{\Psi_1}$, with $\mathcal{B}_1$ being the algebra generated by the matrix elements $(b^i{}_j)^{i=1,2}_{j=1,2}$ and I_{Ψ_1} is the two-sided ideal generated by the commutation relations which correspond to the upper block $(b^i{}_j)^{i=1,2}_{j=1,2}$. In analogy $\mathcal{B}_{\Psi_2} = \mathcal{B}_2/I_{\Psi_2}$ is the quotient algebra related to the lower block $(b^{i+2}{}_{j+2})^{i=1,2}_{j=1,2}$ and its commutation relations. Note, in particular, that in our case $\Psi_1 = \Psi_2 = \pi$.

By requiring multiplicativity, we now extend the coaction maps δ_i to the braided exterior algebras of $\mathcal{S}_i$, by means of

$$\delta_i^{\wedge} : \mathcal{S}_i^{\wedge} \to \mathcal{B}_{\Psi_i} \otimes \mathcal{S}_i^{\wedge}. \tag{57}$$

In particular we have

$$\delta_i^{\wedge}(\zeta_i) = \Delta_i \otimes \zeta_i, \tag{58}$$

where ζ_i is the volume element in $\mathcal{S}_i^{\wedge 2}$, and $\Delta_i \in \mathcal{B}_{\Psi_i}$ is the braided determinant.

By direct calculation it is easy to show that

$$\Delta_1 = b^1{}_1 b^2{}_2 - b^1{}_2 b^2{}_1, \qquad \Delta_2 = b^3{}_3 b^4{}_4 - b^3{}_4 b^4{}_3. \tag{59}$$

Furthermore, since $\mathbf{x}$ and $\mathbf{y}$ are unit vectors, i.e., $\langle \mathbf{x}, \mathbf{x}\rangle = x'^1 x^1 + x'^2 x^2 = 1$, $\langle \mathbf{y}, \mathbf{y}\rangle = y'^1 y^1 + y'^2 y^2 = 1$, it turns out, making use of (46), that each matrix block has quantum determinant equal to 1:

$$\Delta_1 = \Delta_2 = 1. \tag{60}$$

It also follows from (58) and (60), that the extension of the coaction map δ to the total braided algebra $\mathcal{S}^{\wedge}$ results in

$$\delta^{\wedge}(\zeta_1 \wedge \zeta_2) = \delta_1^{\wedge}(\zeta_1) \wedge \delta_2^{\wedge}(\zeta_2) = (\Delta_1 \otimes \zeta_1) \wedge (\Delta_2 \otimes \zeta_2) = 1 \otimes (\zeta_1 \wedge \zeta_2), \tag{61}$$

that is $\Delta(b^{\alpha}{}_{\beta}) = 1$. Consequently our braided group corresponds to a deformation of the group $SL(4, \mathbb{C})$. Note that even though each block matrix has classical characteristics (its elements commute and it has a classical determinant), it cannot be considered as a "classical" $SL(2, \mathbb{C})$, because its matrix entries belong to a non-commutative braided Hopf algebra, whereby the entries belonging to one block do not commute with the entries belonging to the other block.

Another interesting property of our braided group is that it leaves invariant the fundamental quadric $\langle \mathbf{x}, \mathbf{x}\rangle$. This result is an immediate consequence of (38) and the fact that, due to (28)–(31) and their commutation relations (15), the fundamental quadric can be written as

$$\langle \mathbf{x}, \mathbf{x}\rangle = -\tfrac{1}{4}(\psi, \tilde{\psi})(\varphi, \tilde{\varphi}). \tag{62}$$

4.2. Coproduct and coaction maps on $\mathcal{B}$

Recall that in (46) we have derived expressions for the generators of the Hopf algebra $\mathcal{B}$, in terms of the *coordinates* of the underlying (pseudo)-Euclidean space. Note, however, that even though these particular elements of the quadratic algebra of the coordinates form a bialgebra, this does not imply that the vector algebra of the coordinates itself is a bialgebra. In fact no such a structure can be imposed on the coordinates that would be compatible with the coproduct on $\mathcal{B}$ in our construction.

On the other hand these *coordinates*, which represent in the non-deformed case unit vectors defining normal hyperplanes of reflection can, by (28)–(31), be expressed as spinor bilinears. Thus the elements $b^{\alpha}{}_{\beta}$ of the braided matrix can, in turn, be expressed as polynomial functions of the spinor algebra $\mathfrak{S}$. Consequently, the $b^{\alpha}{}_{\beta}$ should also have the structure of a comodule and admit a coaction map, induced from the coaction map on the spinor space. Furthermore this induced map should be compatible with the coproduct map already existent in the bialgebra structure of $\mathcal{B}$. It is the purpose of this section to display the relations between the coproduct and coaction on $\mathcal{B}$, as well as to determine the braided group of which the generators $b^{\alpha}{}_{\beta}$ are a comodule.

To begin, we make use of (46), (42), (28)–(31) and (34) to write

$$
\begin{aligned}
b^1{}_1 &= \tfrac{1}{4}[(\psi,\zeta)\tilde{\varphi}^1\tilde{\eta}^2 + (\psi,\tilde{\zeta})\tilde{\varphi}^1\eta^2 + (\tilde{\psi},\zeta)\varphi^1\tilde{\eta}^2 + (\tilde{\psi},\tilde{\zeta})\varphi^1\eta^2],\\
b^1{}_2 &= -\tfrac{1}{4}[(\psi,\zeta)\tilde{\varphi}^1\tilde{\eta}^1 + (\psi,\tilde{\zeta})\tilde{\varphi}^1\eta^1 + (\tilde{\psi},\zeta)\varphi^1\tilde{\eta}^1 + (\tilde{\psi},\tilde{\zeta})\varphi^1\eta^1],\\
b^2{}_1 &= \tfrac{1}{4}[(\psi,\zeta)\tilde{\varphi}^2\tilde{\eta}^2 + (\psi,\tilde{\zeta})\tilde{\varphi}^2\eta^2 + (\tilde{\psi},\zeta)\varphi^2\tilde{\eta}^2 + (\tilde{\psi},\tilde{\zeta})\varphi^2\eta^2],\\
b^2{}_2 &= -\tfrac{1}{4}[(\psi,\zeta)\tilde{\varphi}^2\tilde{\eta}^1 + (\psi,\tilde{\zeta})\tilde{\varphi}^2\eta^1 + (\tilde{\psi},\zeta)\varphi^2\tilde{\eta}^1 + (\tilde{\psi},\tilde{\zeta})\varphi^2\eta^1],\\
b^3{}_3 &= \tfrac{1}{4}[(\varphi,\eta)\tilde{\psi}^1\tilde{\zeta}^2 + (\varphi,\tilde{\eta})\tilde{\psi}^1\zeta^2 + (\tilde{\varphi},\eta)\psi^1\tilde{\zeta}^2 + (\tilde{\varphi},\tilde{\eta})\psi^1\zeta^2],\\
b^3{}_4 &= -\tfrac{1}{4}[(\varphi,\eta)\tilde{\psi}^1\tilde{\zeta}^1 + (\varphi,\tilde{\eta})\tilde{\psi}^1\zeta^1 + (\tilde{\varphi},\eta)\psi^1\tilde{\zeta}^1 + (\tilde{\varphi},\tilde{\eta})\psi^1\zeta^1],\\
b^4{}_3 &= \tfrac{1}{4}[(\varphi,\eta)\tilde{\psi}^2\tilde{\zeta}^2 + (\varphi,\tilde{\eta})\tilde{\psi}^2\zeta^2 + (\tilde{\varphi},\eta)\psi^2\tilde{\zeta}^2 + (\tilde{\varphi},\tilde{\eta})\psi^2\zeta^2],\\
b^4{}_4 &= -\tfrac{1}{4}[(\varphi,\eta)\tilde{\psi}^2\tilde{\zeta}^1 + (\varphi,\tilde{\eta})\tilde{\psi}^2\zeta^1 + (\tilde{\varphi},\eta)\psi^2\tilde{\zeta}^1 + (\tilde{\varphi},\tilde{\eta})\psi^2\zeta^1].
\end{aligned}
\tag{63}
$$

In the above expressions ψ, $\tilde{\psi}$ denote the second-type semi-spinors associated with the vector $\mathbf{x}$, and φ, $\tilde{\varphi}$ the first-type semi-spinors also associated with $\mathbf{x}$; while ζ, $\tilde{\zeta}$, and η, $\tilde{\eta}$ denote the second- and first-type semi-spinors associated with the vector $\mathbf{y}$, respectively.

Note in addition, that taking $\mathbf{x}$ and $\mathbf{y}$ to be space-like in the classical limit, we have the following extra constraint relations for the spinor algebra:

$$
\langle \mathbf{x},\mathbf{x}\rangle = -\tfrac{1}{4}(\psi,\tilde{\psi})(\varphi,\tilde{\varphi}) = 1,\ \langle \mathbf{y},\mathbf{y}\rangle = -\tfrac{1}{4}(\zeta,\tilde{\zeta})(\eta,\tilde{\eta}) = 1. \tag{64}
$$

Recalling now from (38) in Section 3 that the spinor bilinear product is invariant under the coaction map, it becomes evident that the elements $b^i{}_j$, $i,j = 1,2$, are semi-spinor bilinears of the first-type, while the elements b^{2+i}_{2+j}, $i,j = 1,2$, are semi-spinor bilinears of the second-type. This in turn implies that the generators of the algebra $\mathcal{B}$ are a comodule vector space of the block-diagonal braided group

$$
\begin{pmatrix} \mathcal{D}^{(1/2,0)} \times \mathcal{D}^{(1/2,0)} & 0 \\ 0 & \mathcal{D}^{(0,1/2)} \times \mathcal{D}^{(0,1/2)} \end{pmatrix}, \tag{65}
$$

where $\mathcal{D}$ are representations of $SL(2,\mathbb{C})$. Note, however, that here the matrix elements of $\mathcal{D}^{(1/2,0)}$ do not commute with the elements of $\mathcal{D}^{(0,1/2)}$.

More specifically, using (56) and (63), we have

$$
\begin{aligned}
\delta(b^1{}_1) &= b^1{}_2b^2{}_2\otimes b^2{}_1 - b^1{}_2b^2{}_1\otimes b^2{}_2 + b^1{}_1b^2{}_2\otimes b^1{}_1 - b^1{}_1b^2{}_1\otimes b^1{}_2,\\
\delta(b^1{}_2) &= b^1{}_2b^1{}_1\otimes b^2{}_2 + b^1{}_1b^1{}_1\otimes b^1{}_2 - b^1{}_2b^1{}_2\otimes b^2{}_1 - b^1{}_1b^1{}_2\otimes b^1{}_1,\\
\delta(b^2{}_1) &= b^2{}_2b^2{}_2\otimes b^2{}_1 - b^2{}_2b^2{}_1\otimes b^2{}_2 + b^2{}_1b^2{}_2\otimes b^1{}_1 - b^2{}_1b^2{}_1\otimes b^1{}_2,\\
\delta(b^2{}_2) &= b^2{}_2b^1{}_1\otimes b^2{}_2 + b^2{}_1b^1{}_1\otimes b^1{}_2 - b^1{}_2b^2{}_2\otimes b^2{}_1 - b^2{}_1b^1{}_2\otimes b^1{}_1,\\
\delta(b^3{}_3) &= b^3{}_4b^4{}_4\otimes b^4{}_3 - b^3{}_4b^4{}_3\otimes b^4{}_4 + b^3{}_3b^4{}_4\otimes b^3{}_3 - b^3{}_3b^4{}_3\otimes b^3{}_4,\\
\delta(b^3{}_4) &= b^3{}_4b^3{}_3\otimes b^4{}_4 + b^3{}_3b^3{}_3\otimes b^3{}_4 - b^3{}_4b^3{}_4\otimes b^4{}_3 - b^3{}_3b^3{}_4\otimes b^3{}_3,
\end{aligned}
\tag{66}
$$

$$\delta(b^4{}_3) = b^4{}_4 b^4{}_4 \otimes b^4{}_3 - b^4{}_4 b^4{}_3 \otimes b^4{}_4 + b^4{}_3 b^4{}_4 \otimes b^3{}_3 - b^4{}_3 b^4{}_3 \otimes b^3{}_4,$$
$$\delta(b^4{}_4) = b^4{}_4 b^3{}_3 \otimes b^4{}_4 + b^4{}_3 b^3{}_3 \otimes b^3{}_4 - b^3{}_4 b^4{}_4 \otimes b^4{}_3 - b^4{}_3 b^3{}_4 \otimes b^3{}_3.$$

Using the symplectic spinor metric

$$\omega_{12} = \omega^{12} = -1, \qquad \omega_{34} = (\omega^{34})^{-1} = -\sqrt{\mu}, \tag{67}$$

Eqs. (66) can be summarized in the form

$$\delta(b^\alpha{}_\beta) = b^\alpha{}_\gamma b_\beta^\lambda \otimes b^\gamma{}_\lambda. \tag{68}$$

This last expression exhibits explicitly the induced comodule structure on the generators of the bialgebra $\mathcal{B}$, together with the fact that the indices in $b^\alpha{}_\beta$ transform as spinor indices.

As a final remark observe that using the coproduct structure in $\mathcal{B}$, we can rewrite Eqs. (66) as

$$\delta(b^1{}_1) = b^2{}_2 \phi(b^1{}_1) - b^2{}_1 \phi(b^1{}_2), \tag{69}$$
$$\delta(b^1{}_2) = b^1{}_1 \phi(b^1{}_2) - b^1{}_2 \phi(b^1{}_1), \tag{70}$$
$$\delta(b^2{}_1) = b^2{}_2 \phi(b^2{}_1) - b^2{}_1 \phi(b^2{}_2), \tag{71}$$
$$\delta(b^2{}_2) = b^1{}_1 \phi(b^2{}_2) - b^1{}_2 \phi(b^2{}_1), \tag{72}$$
$$\delta(b^3{}_3) = b^4{}_4 \phi(b^3{}_3) - b^4{}_3 \phi(b^3{}_4), \tag{73}$$
$$\delta(b^3{}_4) = b^3{}_3 \phi(b^3{}_4) - b^3{}_4 \phi(b^3{}_3), \tag{74}$$
$$\delta(b^4{}_3) = b^4{}_4 \phi(b^4{}_3) - b^4{}_3 \phi(b^4{}_4), \tag{75}$$
$$\delta(b^4{}_4) = b^3{}_3 \phi(b^4{}_4) - b^3{}_4 \phi(b^4{}_3). \tag{76}$$

These are our desired relations between the coaction and the coproduct maps on $\mathcal{B}$.

4.3. The braided $\underline{S}pin(4-h, h)$ groups from the braided $\mathcal{B}_\Psi$ group

To conclude this section, we now show how the braided spin group $\underline{S}pin(4-h, h)$, given by the set $\{s \in Cl(W, \tau), dim(W) = 4 | \; s = 1, \; B(\mathbf{x}_1, \mathbf{x}_2), \; B(\mathbf{x}_1, \mathbf{x}_2)B(\mathbf{x}_3, \mathbf{x}_4), \mathbf{x}_i = \mathbf{x}_i^* \in W, |\mathbf{x}_i| = 1\}$, is constructed from the braided group $\mathcal{B}_\Psi$ derived above. The procedure for this construction is contained in the following:

Proposition. *The algebra of $\underline{S}pin(4-h, h)$ belongs to the braided group $\mathcal{B}_\Psi$ with matrix representation*

$$\begin{pmatrix} \prod_i B_1^i & 0 \\ 0 & \prod_i B_2^i \end{pmatrix} \in SL(2, \mathbb{C}) \underline{\times} SL(2, \mathbb{C}),$$

with commutation relations for the entries given by (47), *and representation space $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, where* $\dim \mathcal{S}_1 = \dim \mathcal{S}_2 = 2$.

In the above we have used the symbol $\underline{\times}$ to emphasize the fact that between the matrix entries of the first and second element of the Cartesian product, there exist non-trivial commutation relations.

Proof. We already have shown that each pair $B(\mathbf{x}_i, \mathbf{x}_{i+1})$ belongs to a braided group with matrix representation

$$\begin{pmatrix} B_1^i & 0 \\ 0 & B_2^i \end{pmatrix} \text{ of } SL(2,\mathbb{C})\underline{\times}SL(2,\mathbb{C}).$$

Hence s has a matrix representation

$$s = \begin{pmatrix} \prod_i B_1^i & 0 \\ 0 & \prod_i B_2^i \end{pmatrix} \quad \text{of } SL(2,\mathbb{C})\underline{\times}SL(2,\mathbb{C}).$$

The above is due to the already mentioned fact that each block behaves classically and, therefore, $\det(B_j^1 B_j^3 \cdots B_j^k) = \prod_i \det(B_j^i) = 1$, for $j = 1, 2$. In addition, it can be shown by induction that the commutation relations for the above product matrix are those given by (47). Finally observe that the matrix elements $s^\alpha{}_\beta$ are polynomials in the generators $b^\alpha{}_\beta$. Consequently they belong to the braided Hopf algebra $\mathcal{B}_\Psi$. □

Note in fact that the braided groups $\underline{S}pin(4-h, h)$ actually have the ordinary group properties. Indeed, it is a simple matter to verify that they satisfy the group closure relation and also for any generating matrix of $\underline{S}pin(4-h, h)$, $s^{-1} = 1$, $B(\mathbf{x}_2, \mathbf{x}_1)$, $B(\mathbf{x}_4, \mathbf{x}_3)B(\mathbf{x}_2, \mathbf{x}_1)$.

In the classical case $S^3 \times S^3 \simeq \mathcal{S}pin(4-h, h)$, and therefore every element $s \in \mathcal{S}pin(4-h, h)$ is associated with a transformation $B(\mathbf{x}, \mathbf{y})$, where $\langle \mathbf{x}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle = \mathbf{1}$. However, for the case $\mu \neq 1$, where we lack the concept of quaternions to be able to define the 2 to 1 homomorphism between $S^3 \times S^3$ and $SO(4-h, h)$, and we also lack the concept of a unique double covering of $SO(4-h, h)$ by $\mathcal{S}pin(4-h, h)$, it becomes difficult to conclude that $S^3 \times S^3$ and $\mathcal{S}pin(4-h, h)$ are isomorphic.

5. Relation between the braided spin and the orthogonal groups

5.1. The $SO(4-h, h)$ braided groups

As we have shown in Section 4, the *coordinates* of the underlying (pseudo)-Euclidean space can be expressed as spinor bivectors. Thus the coaction map on $\mathcal{S}$ induces a coaction map on the generators of the algebra $\mathcal{A}_{\hat{R}}$, given by $\delta : (x^\alpha) \mapsto t^\alpha{}_\beta \otimes x^\beta$.

Applying the respective coactions on both sides of (28)–(31) and comparing coefficients, we get

$$\begin{aligned}
t^1{}_1 &= b^3{}_3 b^2{}_2, & t^2{}_1 &= b^4{}_3 b^2{}_2, \\
t^1{}_2 &= b^3{}_4 b^2{}_2, & t^2{}_2 &= b^4{}_4 b^2{}_2, \\
t^1{}_{1'} &= -\mu^{-1} b^3{}_4 b^2{}_1, & t^2{}_{1'} &= -\mu^{-1} b^4{}_4 b^2{}_1, \\
t^1{}_{2'} &= b^3{}_3 b^2{}_1, & t^2{}_{2'} &= b^4{}_3 b^2{}_1, \\
t^{1'}{}_1 &= -\mu^{-1} b^4{}_3 b^1{}_2, & t^{2'}{}_1 &= b^3{}_3 b^1{}_2, \\
t^{1'}{}_2 &= -\mu b^4{}_4 b^1{}_2, & t^{2'}{}_2 &= \mu^2 b^3{}_4 b^1{}_2, \\
t^{1'}{}_{1'} &= b^4{}_4 b^1{}_1, & t^{2'}{}_{1'} &= -\mu b^3{}_4 b^1{}_1, \\
t^{1'}{}_{2'} &= -\mu^{-1} b^4{}_3 b^1{}_1, & t^{2'}{}_{2'} &= b^3{}_3 b^1{}_1.
\end{aligned} \tag{77}$$

In this way we obtain a relation between the matrix $T = (t^{\alpha}_{\beta})$ and the braided matrix B, discussed in Section 4, via the product $B_1 \underline{\otimes} B_2 \in SL(2, \mathbb{C}) \underline{\otimes} SL(2, \mathbb{C})$; where B_1 and B_2 represent the upper and lower blocks of the whole block-diagonal matrix B.

Note that B_1 and B_2 are, respectively, elements of the braided Hopf algebras $\mathcal{B}_{\Psi_1}$ and $\mathcal{B}_{\Psi_2}$, with a matrix representation given by $SL(2, \mathbb{C})$. Taking into account the braid operator Ψ, defined by relations (47), it follows that these Hopf algebras belong to a braided monoidal category. It also follows (cf. [8]) that the product $\mathcal{B}_{\Psi_1} \underline{\otimes} \mathcal{B}_{\Psi_2}$ is again a braided Hopf algebra, constructed on $\mathcal{B}_{\Psi_1} \otimes \mathcal{B}_{\Psi_2}$, with a product $(\cdot_{\mathcal{B}_{\Psi_1}} \otimes \cdot_{\mathcal{B}_{\Psi_2}}) \Psi_{\mathcal{B}_{\Psi_2}, \mathcal{B}_{\Psi_1}}$, and belonging to the same category. Furthermore in our specific case this product inherits the structure of a bialgebra from $\mathcal{B}_{\Psi}$.

In conclusion, $\mathcal{B}_{\Psi_1} \underline{\otimes} \mathcal{B}_{\Psi_2}$ is a braided group, with matrix representation

$$B_1 \underline{\otimes} B_2 \in SL(2, \mathbb{C}) \underline{\otimes} SL(2, \mathbb{C})$$

and representation space $\mathcal{W} = \mathcal{S}_1 \underline{\otimes} \mathcal{S}_2$, coacting on the *coordinates* x^{α} according to

$$\delta(x^{\alpha}) = t^{\alpha}_{\beta} \otimes x^{\beta}. \tag{78}$$

We next analyze the *-structure of the algebra $\mathcal{T}_R$ inherited from the one defined on the generators of the braided $Spin$ group. Specifically for the Minkowski case and an isotropic basis, this follows from (49). Thus

$$(t^1_1)^* = t^{1'}_{1'}, \qquad (t^1_2)^* = \mu^{-1} t^{1'}_{2}, \tag{79}$$

$$(t^1_{1'})^* = t^{1'}_{1}, \qquad (t^1_{2'})^* = \mu t^{1'}_{2'}, \tag{80}$$

$$(t^2_1)^* = \mu t^2_{1'}, \qquad (t^2_2)^* = t^2_2, \tag{81}$$

$$(t^2_{2'})^* = t^2_{2'}, \qquad (t^{2'}_1)^* = \mu^{-1} t^{2'}_{1'}, \tag{82}$$

$$(t^{2'}_2)^* = t^{2'}_2, \qquad (t^{2'}_{2'})^* = t^{2'}_{2'}. \tag{83}$$

It can be verified that the *-structure given by the above relations satisfies the conditions:

$$*m\delta(x^{\alpha}) = m\pi(* \otimes *)\delta(x^{\alpha}), \qquad \phi((b^{\alpha}_{\beta})^*) = (* \otimes *)\pi \Psi^{-1} \phi(b^{\alpha}_{\beta}),$$

and is therefore compatible with the coaction and coproduct maps.

To complete our construction, in the following we obtain explicit expressions for the quantum determinant and antipode of the matrix T.

5.2. *Braided determinant and antipode of T*

Let $\mathcal{W}$ denote the vector space generated by the coordinates x^{α}, and $\mathcal{T}$ the non-commutative algebra generated by the matrix elements t^{α}_{β}. Also let $\delta^{\wedge} : \mathcal{W}^{\wedge} \to \mathcal{T} \otimes \mathcal{W}^{\wedge}$ be the natural multiplicative extension of the coaction map δ given by (78). In particular, we have

$$\delta^{\wedge}(\omega) = \Delta \otimes \omega, \tag{84}$$

where ω is the volume element in $\mathcal{W}^{\wedge 4}$, and $\Delta \in \mathcal{T}$ is the braided determinant. From the comodule properties we have

$$\phi(\Delta) = \Delta \otimes \Delta, \quad \epsilon(\Delta) = 1.$$

Assuming that $\mathcal{W}^{\wedge}$ is an immersion in $\mathcal{W}^{\otimes}$, and making use of the braid operator $\hat{R}$, we have

$$\omega = \sum_{\alpha} x^{\alpha} \otimes s^{\alpha}, \tag{85}$$

where $\{s^{\alpha}\} \in \mathcal{W}^{\wedge 3}$ is a basis for this space.

Since as vector spaces $\mathcal{W}^{\wedge 3} \simeq \mathcal{W}$ (their dimensions are the same), we can write

$$\delta^{\wedge}(s^{\alpha}) = \sum_{\beta} \tilde{t}^{\alpha}_{\ \beta} \otimes s^{\beta}, \tag{86}$$

where $(\tilde{t}^{\alpha}_{\ \beta}) \in M_4(\tilde{\mathcal{T}})$, with $\tilde{\mathcal{T}}$ being the algebra obtained from adding to $\mathcal{T}$ the inverse of Δ. It can be seen from the coproduct and counit relations for the determinant, that ϕ and ϵ admit natural extensions to $\tilde{\mathcal{T}}$ given by $\phi(\tilde{t}^{\alpha}_{\ \beta}) = \sum_{\gamma} \tilde{t}^{\alpha}_{\ \gamma} \otimes \tilde{t}^{\gamma}_{\ \beta}$, $\epsilon(\tilde{t}^{\alpha}_{\ \beta}) = \delta^{\alpha}_{\ \beta}$.

Consider now the scalar matrix S given by

$$x^{\alpha} \wedge s^{\beta} = S^{\alpha\beta}\, \omega. \tag{87}$$

By direct calculation we obtain

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mu^{-1} & 0 & 0 \\ 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{88}$$

Thus applying relations (84)–(87), we arrive at

$$\Delta I = \Psi(\tilde{T})^{\top} T, \tag{89}$$

$$\Delta I = T S \Psi(\tilde{T})^{\top} S^{-1}, \tag{90}$$

where Ψ symbolizes the braid operator between the matrix elements of T and the coordinates x^{α}, while $T^{\top}$ is the ordinary matrix transpose.

From (89) and (90) it follows that the antipode $\kappa(T) \in M_4(\tilde{\mathcal{T}})$ is given by

$$\kappa(T) = \Delta^{-1}\Psi(\tilde{T})^{\top} = S\Psi(\tilde{T})^{\top} S^{-1} \Delta^{-1}. \tag{91}$$

So far we have made no use of the construction of the matrix T as a function of B. If we use (84) to calculate the determinant explicitly, we get $\Delta \otimes \omega = \delta^{\wedge}(\omega) = \delta^{\wedge}(A_4(x^1 \otimes x^2 \otimes x'^1 \otimes x'^2)) = \sum_{\pi \in S_4} (-1)^{l(\pi)} \sigma_{\pi}(\delta(x^1) \otimes \delta(x^2) \otimes \delta(x'^1) \otimes \delta(x'^2))$, where A_4 is the anti-symmetrizer in $\mathcal{W}^{\otimes 4}$. Carrying out the coaction maps and comparing terms on both sides of the resulting equation, yields

$$\begin{aligned}\Delta = {} & t^{1}{}_{1}t^{2}{}_{2}t^{1'}{}_{1'}t^{2'}{}_{2'} - t^{1}{}_{2}t^{2}{}_{1}t^{1'}{}_{1'}t^{2'}{}_{2'} - t^{1}{}_{2'}t^{2}{}_{2}t^{1'}{}_{1'}t^{2'}{}_{1} + t^{1}{}_{2}t^{2}{}_{1'}t^{1'}{}_{1}t^{2'}{}_{2'} \\ & - t^{1}{}_{2}t^{2}{}_{1'}t^{1'}{}_{2'}t^{2'}{}_{1} + t^{1}{}_{1'}t^{2}{}_{2}t^{1'}{}_{2'}t^{2'}{}_{1} - \mu^{2}t^{1}{}_{1'}t^{2}{}_{2'}t^{1'}{}_{2}t^{2'}{}_{1} + \mu^{-2}t^{1}{}_{2'}t^{2}{}_{1'}t^{1'}{}_{2}t^{2'}{}_{1} \\ & + t^{1}{}_{2}t^{2}{}_{1}t^{1'}{}_{2'}t^{1'}{}_{2'}t^{2'}{}_{1'} + \mu^{-2}t^{1}{}_{2'}t^{2}{}_{2}t^{1'}{}_{1}t^{2'}{}_{1'} + \mu^{4}t^{1}{}_{1'}t^{2}{}_{2'}t^{1'}{}_{1}t^{2'}{}_{2} - t^{1}{}_{2'}t^{2}{}_{1'}t^{1'}{}_{1}t^{2'}{}_{2} \\ & + t^{1}{}_{2'}t^{2}{}_{1}t^{1'}{}_{1'}t^{2'}{}_{2} - \mu^{-2}t^{1}{}_{2'}t^{2}{}_{1}t^{1'}{}_{2}t^{2'}{}_{1'} - t^{1}{}_{1}t^{2}{}_{1'}t^{1'}{}_{2}t^{2'}{}_{2'} + t^{1}{}_{1}t^{2}{}_{1'}t^{1'}{}_{2'}t^{2'}{}_{2} \\ & - t^{1}{}_{1}t^{2}{}_{2}t^{1'}{}_{2'}t^{2'}{}_{1'} + \mu^{-2}t^{1}{}_{2}t^{2}{}_{2'}t^{1'}{}_{1'}t^{2'}{}_{1} - t^{1}{}_{2}t^{2}{}_{2'}t^{1'}{}_{1}t^{2'}{}_{1'} - t^{1}{}_{1'}t^{2}{}_{2}t^{1'}{}_{1}t^{2'}{}_{2'} \\ & - \mu^{-2}t^{1}{}_{1}t^{2}{}_{2'}t^{1'}{}_{1'}t^{2'}{}_{2} + t^{1}{}_{1}t^{2}{}_{2'}t^{1'}{}_{2}t^{2'}{}_{1'} + \mu^{-2}t^{1}{}_{1'}t^{2}{}_{1}t^{1'}{}_{2}t^{2'}{}_{2'} - \mu^{-2}t^{1}{}_{1'}t^{2}{}_{1}t^{1'}{}_{2'}t^{2'}{}_{2}.\end{aligned} \tag{92}$$

Finally writing the above relation in terms of b^{α}_{β}, we obtain

$$\Delta = (\Delta_1)^2 = 1. \tag{93}$$

This result establishes the fact that the braided group T preserves the volume element (since $\delta^{\wedge}(w) = 1 \otimes w$) and, consequently, it can be identified with a deformation of the group $SL(4, \mathbb{C})$. Also note that when substituting (93) in (91) shows that the inverse of the matrix T is its transpose, with some deformation given by the braid Ψ. This suggests that T is related to a deformation of $SO(4-h, h)$. To establish this fact, we have only to verify that T leaves invariant the fundamental quadric of the space $\mathcal{W}$. But we already have shown in Section 4 that this fundamental quadric is invariant under the coaction (44) of B. Thus, since T is determined by the compatibility relations (77) between the coaction maps for both braided groups, it follows that the fundamental quadric $\langle \mathbf{x}, \mathbf{x} \rangle$ is invariant under the coaction (78) of T.

Hence

$$\langle \mathbf{x}, \mathbf{x} \rangle = x^{\alpha} x^{\beta} M_{\alpha\beta} = \delta(\langle \mathbf{x}, \mathbf{x} \rangle) = (t^{\alpha}_{\gamma} \otimes x^{\gamma})(t^{\beta}_{\lambda} \otimes x^{\lambda}) M_{\alpha\beta},$$

where

$$M = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

is the metric for the space $\mathcal{W}$. Using now the braiding

$$x^{\gamma} t^{\beta}_{\lambda} = \Psi^{\gamma\beta}{}_{\lambda\rho}{}^{\sigma}{}_{\mu}\, t^{\rho}_{\sigma} x^{\mu}, \tag{94}$$

we get

$$t^{\alpha}_{\gamma} M_{\alpha\beta} \Psi^{\gamma\beta}{}_{\lambda\rho}{}^{\sigma}{}_{\mu}\, t^{\rho}_{\sigma} = M_{\mu\lambda}. \tag{95}$$

Eq. (95) is the equivalent of orthogonality for our braided T-group.

To sum up, the algebra $\mathcal{T}$ generated by the matrix elements t^{α}_{β} determined by relations (77) satisfies the axioms of a braided group with an antipode which turns out to be its braided transpose and a determinant which is equal to 1. We thus have the following:

Definition. The braided special orthogonal group $\underline{SO}(4-h, h)$ is the braided Hopf algebra $\mathcal{T} = \mathcal{B}_{\Psi_1} \underline{\otimes} \mathcal{B}_{\Psi_2}$, with a *-structure induced by the one for the *coordinates*, given in (53).

Note that in our particular case $\mathcal{T}$ is actually also a group. Indeed, let $T, T' \in \mathcal{T}$, then $T \cdot T' = (B_1 B_1' \otimes B_2 B_2')\Psi_{B_2, B_1}$. But the matrix elements of $B_1 B_1'$ and $B_2 B_2'$ satisfy the commutation relations (47), which in turn implies that $T \cdot T' \in \mathcal{T}$.

We thus have found a morphism which relates $\underline{SO}(4-h, h)$ to the braided group $\underline{Spin}$ and, when restricted to the matrix representation for which the entries are the generators, is given by

$$F : SL(2, \mathbb{C})\underline{\times}SL(2, \mathbb{C}) \to SL(2, \mathbb{C})\underline{\otimes}SL(2, \mathbb{C}), \qquad F(B_1 \underline{\times} B_2) = B_1 \underline{\otimes} B_2.$$

Note that F is not injective, since B and $-B$ give the same matrix T. However, it turns out to be surjective by definition; in other words $F(SL(2, \mathbb{C})\underline{\times}SL(2, \mathbb{C})) = \underline{SO}(4, \mathbb{C})$.

If we take the solution (50) and (51), corresponding to the *-operation on the matrix elements of B for the Euclidean metric, one can check that $B_i^{-1} = (B_i^{\top})^*$ for $i = 1, 2$. In this case, the morphism F reduces to

$$F(SU(2)\underline{\times}SU(2)) = \underline{SO}(4).$$

On the other hand, if we take the solution (49), which corresponds to the Minkowski metric, the matrix blocks B_1 and B_2 are not independent, and the morphism F can be restricted to a single matrix block:

$$F(SL(2, \mathbb{C})) = \underline{SO}(3, 1).$$

Note that in the classical limit the usual relations are recovered, with the local isomorphisms:

$$SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \sim SO(4, \mathbb{C}),$$
$$SU(2) \times SU(2) \sim SO(4),$$
$$SL(2, \mathbb{C}) \sim SO(3, 1).$$

These relations, however, do not generalize easily to the deformed case, since one would need a concept similar to that of a simply connected covering space. The fact that the deformed groups are not groups, due to the failure of the closure conditions, prevents a definition of the concept of continuous transformations that would take one point in space to another via a succession of infinitesimal transformations. Consequently, the term *simply connected* loses its usual meaning in deformed geometries. In our case, the closure relation is satisfied. The difficulty appears when one tries to define a neighborhood of the identity, and over this neighborhood, a homeomorphism to the real space to obtain the concept of continuous and differentiable coordinate functions. A manifold structure has yet to be defined in deformed geometrical theories. One possibility is contained in Connes work [12], which relates stability and holomorphic functional calculus with a generalization of smoothness.

6. Summary and conclusions

Starting from the natural decomposition of a (pseudo)-Euclidean space into two isotropic subspaces V and V' of equal dimension, we constructed in [9] a general q-deformed Clifford algebra as a braided monoidal category of bicovariant bimodules which incorporates, in the quantum context, the Cartan theory of spinors. The class of braidings involved in this construction is of the Hecke type. Because of the nonlinearity and interpretation difficulties of the derivation operators in the *-compatible differential calculus resulting from such, or more complicated intertwiners, we were led in [5] to consider involutive braids (diagonal R-matrices). Although simple, involutive braids are not trivial, and yield the usual interpretation of differentials as shifts of coordinates with left and right actions of derivation operators occurring as two representations of the same abstract operator. Moreover, from a mathematical point of view, very interesting purely quantum phenomena appear already at the level of such diagonal R-matrices, as for example the possible deviations (from its classical counterpart) in the Poincaré series of the braided exterior algebra. Hence, observing that the same algebra for the generators of the q-Clifford algebra $Cl(V, \sigma)$, associated to the isotropic subspace V, can be obtained by considering the involutive braid τ, and further using the natural functoriality of the contraction $\langle\ ,\ \rangle$ of V with V', the remaining intertwiners were derived in [9] in order to complete the Clifford algebra $Cl(\tau, W)$ for the total space $W = V \oplus V'$ (cf. Eqs. (11)–(14)). Furthermore, requiring that the fundamental property of spinor transformations (cf. Eq. (39)) be preserved in the quantum case, a non-commutative algebra $\mathcal{A}$ was obtained for the coordinates of the underlying (pseudo)-Euclidean spaces, given by (15) or, equivalently, by the block-diagonal homomorphism $\hat{R}$ in (17). We then showed that a natural deformation of the classical Cartan approach to construct spin groups, produces braided spin and Lorentz groups instead of quantum groups. The above outline helps to further stress some of the similarities as well as the basic differences between some aspects of our approach and others appearing in the literature. Specifically, our involutive block-diagonal $\hat{R}$-matrices and $\mathcal{A}$ algebra for the deformed q-coordinates (in four-dimensional space) are essentially the same as those used by Chaichian et al. (cf. [3]) and references therein) to study quantum Lorentz and Poincaré groups; they differ, on the other hand, substantially from the braidings used by Carow-Watamura et al. (cf. [2])in their approach to quantum $Sl(2, \mathbb{C})$ and Lorentz groups. Indeed, in the work of the latter authors the braid matrix $\hat{R}$ satisfies a cubic characteristic equation and it is not, therefore, in the Hecke class. Also, in the construction of our braided spin groups we use Cartan spinors (which are essentially Dirac spinors, modulo a constant complex matrix transformation) to express the q-coordinates which generate the $\mathcal{A}$ algebra as a tensor product of such spinors. The braiding for the spin groups and the algebra of these spinor components are both then univocally determined by the requirement of invariance of the spinor bilinear under the action of the generators of the Clifford algebra (see Eqs. (37)–(42)). Here again our spinor algebra Ξ turns out to be basically the same as the one found in [3], but it is different to the one derived in [2]. Furthermore, our spin groups, being braided groups and not quantum groups are completely different to the ones considered in all the previously referred papers. In fact, if we were to set $\Psi = 1$ then our algebra $\mathcal{B}$ of spinor transformations would not turn out to be a quantum Hopf algebra, since the axioms of the coproduct would be violated. One could still pose the following valid question (and we thank the referee for suggesting it): What would happen if we were to take our involutive choice of $\hat{R}$ matrix and use it in the commutation relations for the spinors obtained by Carow-Watamura et al. We shall consider this question in what follows, and when referring to expressions in [2] we shall consistently mean those in the first paper cited there. To begin with, note that our $\hat{R}$ matrix is a non-diagonal 16×16 matrix whose entries, derivable from (42), we shall denote by $\hat{R}^{\alpha\beta}_{\lambda\kappa}$, with the pairs of upper and lower indices having the range $11, \dots, 22, \tilde{1}1, \dots, \tilde{2}\tilde{2}$. In order to relate with the notation used in [2], we make the identifications

$$z_1 = (z_1^1, z_1^2) = (\varphi^1, \varphi^2), \tag{96}$$

$$\tilde{z}_1 = (z_1^{\tilde{1}}, z_1^{\tilde{2}}) = (\psi^1, \psi^2), \tag{97}$$

$$z_2 = (z_2^1, z_2^2) = (\eta^1, \eta^2), \tag{98}$$

$$\tilde{z}_2 = (z_2^{\tilde{1}}, z_2^{\tilde{2}}) = (\zeta^1, \zeta^2). \tag{99}$$

It then follows that the equations labeled as (II.22), (II.23), and (II.32)–(II.35) in [2] take the form

$$\begin{aligned}
z_1^\alpha z_2^\beta &= \hat{R}^{\alpha\beta}_{\lambda\kappa} z_2^\lambda z_1^\kappa, \quad \alpha, \beta, \lambda, \kappa = 1, 2,\\
z_2^{\tilde{\alpha}} z_1^{\tilde{\beta}} &= \hat{R}^{\tilde{\alpha}\tilde{\beta}}_{\tilde{\lambda}\tilde{\kappa}} z_2^{\tilde{\lambda}} e z_1^{\tilde{\kappa}}, \quad \tilde{\alpha}, \tilde{\beta}, \tilde{\lambda}, \tilde{\kappa} = \tilde{1}, \tilde{2},\\
z_1^\alpha z_2^{\tilde{\beta}} &= k' \hat{R}^{\alpha\tilde{\beta}}_{\tilde{\lambda}\kappa} z_2^{\tilde{\lambda}} z_1^\kappa, \quad \alpha, \kappa = 1, 2;\ \tilde{\beta}, \tilde{\lambda} = \tilde{1}, \tilde{2},\\
z_2^\alpha z_1^{\tilde{\beta}} &= k' \hat{R}^{\alpha\tilde{\beta}}_{\tilde{\lambda}\kappa} z_1^{\tilde{\lambda}} z_2^\kappa, \quad \alpha, \kappa = 1, 2;\ \tilde{\beta}, \tilde{\lambda} = \tilde{1}, \tilde{2},\\
z_1^\alpha z_1^{\tilde{\beta}} &= \frac{1}{k'q} \hat{R}^{\alpha\tilde{\beta}}_{\tilde{\lambda}\kappa} z_1^{\tilde{\lambda}} z_1^\kappa, \quad \alpha, \kappa = 1, 2;\ \tilde{\beta}, \tilde{\lambda} = \tilde{1}, \tilde{2},\\
z_2^\alpha z_2^{\tilde{\beta}} &= \frac{1}{k'q} \hat{R}^{\alpha\tilde{\beta}}_{\tilde{\lambda}\kappa} z_2^{\tilde{\lambda}} z_2^\kappa, \quad \alpha, \kappa = 1, 2;\ \tilde{\beta}, \tilde{\lambda} = \tilde{1}, \tilde{2}.
\end{aligned} \tag{100}$$

It is a rather straightforward matter to verify that the above equations satisfy (42), extended to apply to different Cartan spinors, provided we set $k' = q = 1$ in (100), with $\hat{R}(\mu)$. This result is, in particular, in agreement with our formalism where the planes formed from semispinors of the same type are classical (the components of a semi-spinor commute for underlying spaces of dim $= 4$). In addition, the commutation relations for the spin matrices M and $\tilde{M}$ given in Eq. (II.36) of the same considered paper agree, when using our involutive braid, with the ones we obtained for the quantum spin groups in four-dimensions in [5], and express the fact that the entries of each of the spin matrices commute among themselves although, of course, they do not commute with the entries of the other matrix. Finally, inserting (96)–(99) in (III.24) of [2] for the coordinates, and making use of (42) we obtain (15) for the generating algebra of the coordinates, instead of (III.25) (after making the identifications $A \sim x^1, B \sim x'^1, C \sim x'^2, D \sim x^2$). For a further discussion of our approach to quantum spin groups, and the different resulting commutation relations for the spinor components and quantum spin matrices, depending on the dimensions and metrics considered for the underlying spaces, we refer the reader to the work cited in [5].

Acknowledgements

The authors would like to thank M. Đurđević for interesting and inspiring discussions, on the subject. One of the authors (JDV) is grateful for partial support from grant No. DGAPA-UNAM-IN100694.

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